

SUBJECT: MATHEMATICS

**PAPER: ADVANCED MATHEMATICAL
METHODS**

PAPER CODE:H 3051(PAPER III)

CLASS: MSc III SEMESTER

TAUGHT BY:DR SATISH KUMAR

**ASSOCIATE PROFESSOR
DN COLLEGE MEERUT**

Remember! If $f(x)$ is a fn on $[-\pi, \pi]$ then

① Fourier cosine series is
i.e we assume $f(x)$ is an even fn.
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$, $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$$

② Fourier sine series is
i.e we assume $f(x)$ is an odd function
 $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$

Q1. Find Fourier sine series of
 $f(x) = \begin{cases} x, & 0 < x \leq \frac{\pi}{2} \\ \frac{\pi}{2}, & \frac{\pi}{2} < x \leq \pi \end{cases}$

Solution. we know Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$b_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} f(x) \sin nx dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} x \sin nx dx + \frac{2}{\pi} \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2} \sin nx dx$$

$$= \frac{2}{\pi} \left[-x \frac{\cos nx}{n} \Big|_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} (-1) \cdot \frac{\cos nx}{n} dx \right] + \int_{\frac{\pi}{2}}^{\pi} \sin nx dx$$

$$b_n = -\frac{1}{n} \cos \frac{n\pi}{2} + \frac{2}{\pi n} \int_0^{\frac{\pi}{2}} \cos nx dx - \frac{1}{n} [\cos nx]_{\frac{\pi}{2}}^{\pi}$$

(2) H-3051, Dr. Satish Kumar

$$b_n = -\frac{1}{n} \cos \frac{n\pi}{2} + \frac{2}{\pi n^2} \left[\sin n\pi \right]_0^{\frac{\pi}{2}} - \frac{1}{n} \left[\cos n\pi - \cos \frac{n\pi}{2} \right]$$

$$b_n = -\frac{1}{n} \cos \frac{n\pi}{2} + \frac{2}{\pi n^2} \sin \frac{n\pi}{2} - \frac{1}{n} \cos n\pi + \frac{1}{n} \cos \frac{n\pi}{2}$$

$$b_n = \frac{2}{\pi n^2} \sin \frac{n\pi}{2} - \frac{1}{n} \cos n\pi$$

ii

$$b_1 = \frac{2}{\pi} \sin \frac{\pi}{2} - \cos \pi \Rightarrow \boxed{b_1 = \frac{2}{\pi} + 1}$$

$$b_2 = \frac{2}{4\pi} \cdot 0 - \frac{1}{2} \cos 2\pi \Rightarrow b_2 = -\frac{1}{2}$$

$$b_3 = \frac{2}{9\pi} \sin \frac{3\pi}{2} - \frac{1}{3} \cos 3\pi$$

$$= -\frac{2}{9\pi} + \frac{1}{3}$$

$$\left[\begin{array}{l} \because \sin \frac{3\pi}{2} = -1 \\ \& \cos 3\pi = -1 \end{array} \right]$$

Required f_n is given by

$$\therefore f(x) = \underbrace{\left(\frac{2}{\pi} + 1\right) \sin x}_{or} + \underbrace{\left(-\frac{1}{2}\right) \sin 2x}_{+ \dots} + \underbrace{\left(-\frac{2}{9\pi} + \frac{1}{3}\right) \sin 3x}_{+ \dots}$$

$$f(x) = \left(\frac{2}{\pi} + 1\right) \sin x - \frac{1}{2} \sin 2x + \left(\frac{1}{3} - \frac{2}{9\pi}\right) \sin 3x + \dots$$

(2) Find Fourier sine series of

Ans.

$$f(x) = e^{ax}, \quad 0 < x < \pi$$

Sol. Fourier sine series in $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx = \frac{2}{\pi} \int_0^{\pi} e^{ax} \sin nx dx$$

$$= \frac{2}{\pi} \frac{1}{a^2 + n^2} \left[e^{ax} (a \sin nx - n \cos nx) \right]_0^{\pi}$$

$$b_n = \frac{2}{\pi(a^2 + n^2)} \left[e^{a\pi} (0 - n \cos n\pi) - 1 (0 - n) \right]$$

$$b_n = \frac{2}{\pi(a^2 + n^2)} \left[n - n e^{a\pi} (-1)^n \right]$$

(3) H-3051, Dr Satish Kumar

Change of interval and functions with arbitrary period.

In electrical engineering problems or in physical problems, the period of the function is not always 2π but T or $2c$. This period must be converted to the length 2π . The independent variable x is also to be changed proportionally.

Let the function $f(x)$ be defined on $[-c, c]$. Now we want to change the function to the period 2π so that we can use the formula for a_0, a_n, b_n .

When $2c$ is the period then variable is x
if 1 is the period then variable is $\frac{x}{2c}$

$\therefore$ if 2π is the period then variable will be
$$\frac{x}{2c} \times 2\pi$$
$$= \frac{\pi x}{c}$$

So take new variable

$$l = \frac{\pi x}{c} \Rightarrow x = \frac{cl}{\pi}$$

$\therefore F(l)$ is a function of period 2π .

(5) H-3051, Dr. Satish Kumar

Therefore,

Half range series for the interval $0 < x < l$

are

(1) Fourier cosine series on $0 < x < l$ as

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}, \quad a_0 = \frac{2}{l} \int_0^l f(x) dx$$
$$, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

(2) Fourier sine series on $0 < x < l$ as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$
$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

Question. I. Find Fourier ^{cosine} series of the following

$$f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$$

Solution.

Given $f(x) = \begin{cases} 2x, & 0 < x < 1 \\ 2(2-x), & 1 < x < 2 \end{cases}$

We have

$$f(x) = \frac{1}{2} a_0 + \sum a_n \cos \frac{n\pi x}{l} + \sum b_n \sin \frac{n\pi x}{l}$$

$$, \quad a_0 = \frac{2}{l} \int_0^l f(x) dx, \quad a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l} dx$$

$$, \quad b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx$$

(6) H-3051, Dr. Satish Kumar

Here $c=2$

$$a_0 = \frac{2}{2} \int_0^2 f(x) dx = \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

$$\begin{aligned} a_0 &= \int_0^1 2x dx + \int_1^2 2(2-x) dx \\ &= 1 + 2 \left[2x - \frac{x^2}{2} \right]_1^2 = 1 + 2 \left[(2) - (2 - \frac{1}{2}) \right] \\ &= 1 + 2 \left[2 - 2 + \frac{1}{2} \right] = 2 \Rightarrow \boxed{a_0 = 2} \end{aligned}$$

$$\begin{aligned} a_n &= \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx \\ &= \int_0^1 f(x) \cos \frac{n\pi x}{2} dx + \int_1^2 f(x) \cos \frac{n\pi x}{2} dx \\ &= \int_0^1 2x \cos \frac{n\pi x}{2} dx + \int_1^2 2(2-x) \cos \frac{n\pi x}{2} dx \\ a_n &= 2 \left[\frac{2x}{n\pi} \sin \left(\frac{n\pi x}{2} \right) - \int_0^1 \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \right] + 2 \left[\frac{2(2-x)}{n\pi} \sin \frac{n\pi x}{2} \right]_1^2 \\ &\quad - \int_1^2 \left(-1 \right) \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \\ a_n &= 2 \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2 \pi^2} \left(\cos \frac{n\pi x}{2} \right) \right]_0^1 + 2 \left[0 - \frac{2}{n\pi} \sin \frac{n\pi}{2} \right] - \frac{8}{n^2 \pi^2} \left[\cos \frac{n\pi x}{2} \right]_1^2 \\ a_n &= \frac{4}{n\pi} \sin \frac{n\pi}{2} + \frac{8}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{8}{n^2 \pi^2} - \frac{4}{n\pi} \sin \frac{n\pi}{2} - \frac{8}{n^2 \pi^2} \left[\cos n\pi - \cos \frac{n\pi}{2} \right] \\ &= \frac{4}{n\pi} \sin \frac{n\pi}{2} + \frac{8}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{8}{n^2 \pi^2} - \frac{4}{n\pi} \sin \frac{n\pi}{2} - \frac{8}{n^2 \pi^2} \cos n\pi + \frac{8}{n^2 \pi^2} \cos \frac{n\pi}{2} \\ a_n &= \frac{8 \times 2}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{8}{n^2 \pi^2} - \frac{8}{n^2 \pi^2} \cos n\pi \\ a_1 &= \frac{16}{\pi^2} \times 0 - \frac{8}{\pi^2} - \frac{8}{\pi^2} (1) \end{aligned}$$

Take $b_n = 0$ as we are asked only cosine series

$\therefore f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2}$, a_n is just calculated above.

(4) H-3051,

$$\text{So } F(l) = f\left(\frac{cl}{\pi}\right) = \frac{a_0}{2} + a_1 \cos l + a_2 \cos 2l + \dots \\ + b_1 \sin l + b_2 \sin 2l + \dots$$

where

$$a_0 = \frac{1}{\pi} \int_0^{2\pi} F(l) dl = \frac{1}{\pi} \int_0^{2\pi} f\left(\frac{cl}{\pi}\right) dl$$

$$\text{put } l = \frac{\pi x}{c}$$

$$dl = \frac{\pi}{c} dx$$

$$\therefore a_0 = \frac{1}{\pi} \int_0^{2c} f(x) \cdot \frac{\pi}{c} dx$$

$$\text{or } \frac{cl}{\pi} = x$$

$$\boxed{a_0 = \frac{1}{c} \int_0^{2c} f(x) dx}$$

$$\text{Similarly } a_n = \frac{1}{\pi} \int_0^{2\pi} F(l) \cos nl \, dl = \frac{1}{\pi} \int_0^{2\pi} f\left(\frac{cl}{\pi}\right) \cos nl \, dl$$

$$\Rightarrow a_n = \frac{1}{\pi} \int_0^{2c} f(x) \cdot \frac{\pi}{c} \cos\left(n \cdot \frac{\pi x}{c}\right) dx$$

$$\Rightarrow \boxed{a_n = \frac{1}{c} \int_0^{2c} f(x) \cdot \cos\left(\frac{n\pi x}{c}\right) dx}$$

$$\text{, } x = \frac{cl}{\pi}$$

$$\text{and } \boxed{b_n = \frac{1}{c} \int_0^{2c} f(x) \sin\left(\frac{n\pi x}{c}\right) dx}$$

Exercise

① Obtain Fourier cosine series of

$$f(x) = \sin\left(\frac{\pi x}{l}\right), \quad 0 < x < l$$

$$\text{Hint. } a_0 = \frac{2}{l} \int_0^l \sin\left(\frac{\pi x}{l}\right) dx, \quad a_n = \frac{2}{l} \int_0^l \sin\left(\frac{\pi x}{l}\right) \cos \frac{n\pi x}{l} dx$$

$$\text{Ans: } f(x) = \frac{2}{\pi} - \frac{4}{\pi} \left[\frac{1}{3} \cos \frac{2\pi x}{l} + \frac{1}{15} \cos \frac{4\pi x}{l} + \frac{1}{35} \cos \frac{6\pi x}{l} + \dots \right]$$

② Find Fourier series of

$$f(x) = \begin{cases} \frac{1}{2} + x, & -\frac{1}{2} < x \leq 0 \\ \frac{1}{2} - x, & 0 < x < \frac{1}{2} \end{cases}$$

$$\text{Here } 2c = 1 \Rightarrow c = \frac{1}{2}$$

$$\text{Hint. } a_0 = \frac{1}{c} \int_{-c}^c f(x) dx = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) dx$$

$$a_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) \cos 2n\pi x dx, \quad b_n = 2 \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x) \sin 2n\pi x dx$$

$$\text{Ans } f(x) = \frac{1}{4} + \frac{2}{\pi^2} \left[\frac{\cos 2\pi x}{1^2} + \frac{\cos 6\pi x}{3^2} + \frac{\cos 10\pi x}{5^2} + \dots \right]$$

③ Find the Fourier series of

$$f(x) = \begin{cases} 2, & -2 \leq x \leq 0 \\ x, & 0 < x < 2 \end{cases}$$

$$\text{Hint. Here } 2c = 4 \Rightarrow c = \frac{4}{2} = 2$$

$$a_0 = \frac{1}{c} \int_{-c}^c f(x) dx = \frac{1}{2} \int_{-2}^2 f(x) dx$$

$$a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx, \quad b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx$$

$$\text{Ans. } f(x) = \frac{3}{2} - \frac{4}{\pi^2} \left[\frac{1}{1^2} \cos \frac{\pi x}{2} + \frac{1}{3^2} \cos \frac{3\pi x}{2} + \dots \right] \\ - \frac{2}{\pi} \left[\frac{1}{1} \sin \frac{\pi x}{2} + \frac{1}{2} \sin \frac{2\pi x}{2} + \frac{1}{3} \sin \frac{3\pi x}{2} + \dots \right]$$

(B) 11-2021, Dr. Sahel Khatun

(4) Find the Fourier series of

$$f(x) = \begin{cases} -a & , -c < x < 0 \\ a & , 0 < x < c \end{cases}$$

Ans. $f(x) = \frac{4a}{\pi} \left[\sin \frac{\pi x}{c} + \frac{1}{3} \sin \frac{3\pi x}{c} + \frac{1}{5} \sin \frac{5\pi x}{c} + \dots \right]$

(5) Find half range sine series of

$$f(x) = 2x-1, \quad 0 < x < 1.$$

Ans. $f(x) = -\frac{2}{\pi} \left[\sin 2\pi x + \frac{1}{2} \sin 4\pi x + \frac{1}{3} \sin 6\pi x + \dots \right]$

(6) Find Fourier series of

$$f(x) = \begin{cases} -2 & , -4 < x < -2 \\ x & , -2 < x < 2 \\ 2 & , 2 < x < 4 \end{cases} \quad \begin{array}{l} \text{Here } 2c=8 \\ c=4 \end{array}$$

Ans. $f(x) = \frac{4}{\pi} + \frac{8}{\pi^2} \sin \frac{\pi x}{4} - \frac{2}{\pi} \sin \frac{2\pi x}{4} + \dots$

(7) Find Fourier of $f(x) = (x^2-2)$, $x \in]-2, 2[$.

Ans. $f(x) = -\frac{2}{3} - \frac{16}{\pi^2} \left[\cos \frac{\pi x}{2} - \frac{1}{4} \cos \pi x + \frac{1}{9} \cos \frac{3\pi x}{2} + \dots \right]$

(8) If $f(x) = e^{-x}$, $x \in]-c, c[$

show

$$f(x) = (e^c - e^{-c}) \left\{ \frac{1}{2c} - c \left(\frac{1}{c^2 + \pi^2} \cos \frac{\pi x}{c} - \frac{1}{c^2 + 4\pi^2} \cos \frac{2\pi x}{c} + \dots \right) \right. \\ \left. - \pi \left[\frac{1}{c^2 + \pi^2} \sin \frac{\pi x}{c} - \frac{2}{c^2 + 4\pi^2} \sin \frac{2\pi x}{c} + \dots \right] \right\}.$$

(9) H-5051 Dr. Satish Kumar

State Parseval formula and prove it.

Statement. Let $f(x)$ be a periodic function on $[-c, c]$

then prove

$$\int_{-c}^c f^2(x) dx = c \left[\frac{1}{2} a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

Proof. We know Fourier formula as

$$f(x) = \frac{1}{2} a_0 + \sum a_n \cos \frac{n\pi x}{c} + \sum b_n \sin \frac{n\pi x}{c} \quad \text{--- (1)}$$

Multiplying (1) by $f(x)$ and integrating from

$x = -c$ to $x = c$, we have

$$\int_{-c}^c f^2(x) dx = \frac{1}{2} a_0 \int_{-c}^c f(x) dx + \sum a_n \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx + \sum b_n \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx \quad \text{--- (2)}$$

We know that

$$a_0 = \frac{1}{c} \int_{-c}^c f(x) dx \Rightarrow \int_{-c}^c f(x) dx = c a_0 \quad \text{--- (3)}$$

$$a_n = \frac{1}{c} \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx \Rightarrow \int_{-c}^c f(x) \cos \frac{n\pi x}{c} dx = c a_n \quad \text{--- (4)}$$

$$b_n = \frac{1}{c} \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx \Rightarrow \int_{-c}^c f(x) \sin \frac{n\pi x}{c} dx = c b_n \quad \text{--- (5)}$$

So from equation (2), (3), (4) & (5), we get

$$\begin{aligned} \int_{-c}^c f^2(x) dx &= \frac{1}{2} a_0 [c a_0] + \sum a_n (c a_n) + \sum b_n (c b_n) \\ &= c \left[\frac{1}{2} a_0^2 + \sum a_n^2 + \sum b_n^2 \right] \end{aligned}$$

proved.

Note. ① $\frac{CH-01}{(10) H-3051}$ Dr. Sahen Kumar
 of $0 < x < 2c$ or $-c < x < c$

then $\int_0^{2c} f^2(x) dx = c \left[\frac{a_0^2}{2} + \sum a_n^2 + \sum b_n^2 \right]$

② of $0 < x < c$ (Half range cosine series)

$$\int_0^c f^2(x) dx = \frac{c}{2} \left[\frac{a_0^2}{2} + \sum a_n^2 \right]$$

③ of $0 < x < c$ (Half range sine series)

$$\int_0^c f^2(x) dx = \frac{c}{2} \left[\sum b_n^2 \right]$$

④ R.M.S = $\sqrt{\frac{\int_a^b f^2(x) dx}{b-a}}$

Examples

① Using sine series for $f(x) = 1$ in $0 < x < \pi$

Show $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$

Solu. We know the sine series is

$$f(x) = \sum b_n \sin nx$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx = \frac{2}{\pi} \int_0^{\pi} 1 \cdot \sin nx dx$$

$$= -\frac{2}{\pi n} [\cos nx]_0^{\pi} = -\frac{2}{\pi n} [\cos n\pi - 1]$$

$$b_n = -\frac{2}{\pi n} [(-1)^n - 1]$$

$$b_n = \frac{4}{\pi n} \text{ if } n \text{ is even } \& b_n = 0 \text{ if } n \text{ is odd.}$$

CH-01 . (11) H-3051, Dr Sahen Kumar

ii $f(x) = \frac{a_0}{2} + \sum a_n \cos nx + \sum b_n \sin nx$
where $a_n = 0, a_0 = 0$

$$\Rightarrow 1 = \sum b_n \sin nx$$

$$1 = b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + b_4 \sin 4x + \dots$$

$$1 = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \frac{4}{5\pi} \sin 5x + \dots$$

We know Parseval's formula for sine series is

$$\int_0^\pi f^2(x) dx = \frac{\pi}{2} \left[\sum b_n^2 \right]$$

$$\text{ii } \int_0^\pi 1 \cdot dx = \frac{\pi}{2} [b_1^2 + b_3^2 + b_5^2 + \dots]$$

$$\pi = \frac{\pi}{2} \left[\left(\frac{4}{\pi}\right)^2 + \left(\frac{4}{3\pi}\right)^2 + \left(\frac{4}{5\pi}\right)^2 + \dots \right]$$

$$\Rightarrow 2 = \frac{16}{\pi^2} \left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots$$

Q2. If $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ \pi(2-x), & 1 < x < 2 \end{cases}$

Using half range cosine series, show

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots = \frac{\pi^4}{96}$$

Soln. We know half range cosine series is

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos \frac{n\pi x}{c}, \quad c=2, \text{ Here}$$

$$\text{where } a_0 = \frac{2}{c} \int_0^c f(x) dx = \frac{2}{2} \left[\int_0^1 f(x) dx + \int_1^2 f(x) dx \right]$$

$$= \int_0^1 \pi x dx + \int_1^2 \pi(2-x) dx$$

$$a_0 = \frac{\pi}{2} + \pi \left(2x - \frac{x^2}{2} \right)_1^2 = \frac{\pi}{2} + \pi \left(2 - \frac{3}{2} \right) = \pi$$

(12) H-3051, Dr. Sahish Kumar

$$a_n = \frac{2}{c} \int_0^c f(x) \cos \frac{n\pi x}{c} dx = \frac{2}{2} \int_0^2 f(x) \cos \frac{n\pi x}{2} dx$$

$$a_n = \int_0^1 f(x) \cos \frac{n\pi x}{2} dx + \int_1^2 f(x) \cos \frac{n\pi x}{2} dx$$

$$a_n = \int_0^1 \pi x \cos \frac{n\pi x}{2} dx + \int_1^2 \pi (2-x) \cos \frac{n\pi x}{2} dx$$

$$a_n = \pi \left[x \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^1 - \int_0^1 1 \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \right]$$

$$+ \pi \left[(2-x) \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_1^2 - \int_1^2 (-1) \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \right]$$

$$= \pi \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{2}{n\pi} \left(\frac{2}{n\pi} \right) \cos \frac{n\pi x}{2} \Big|_0^1 \right]$$

$$+ \pi \left[0 - \frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{2}{n\pi} (-) \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_1^2 \right]$$

$$= \pi \left[\frac{2}{n\pi} \sin \frac{n\pi}{2} + \frac{4}{n^2 \pi^2} (\cos \frac{n\pi}{2} - 1) \right]$$

$$+ \pi \left[-\frac{2}{n\pi} \sin \frac{n\pi}{2} - \frac{4}{n^2 \pi^2} (\cos n\pi - \cos \frac{n\pi}{2}) \right]$$

$$= \pi \left[\frac{4}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2 \pi^2} - \frac{4}{n^2 \pi^2} \cos n\pi + \frac{4}{n^2 \pi^2} \cos \frac{n\pi}{2} \right]$$

$$= \pi \left[\frac{8}{n^2 \pi^2} \cos \frac{n\pi}{2} - \frac{4}{n^2 \pi^2} - \frac{4}{n^2 \pi^2} \cos n\pi \right]$$

$$a_n = \frac{4}{n^2 \pi} \left[2 \cos \frac{n\pi}{2} - 1 - \cos n\pi \right]$$

putting $n=1, 2, 3, 4, \dots$, we get

$$a_1 = 0, a_2 = -\frac{4}{\pi}, a_3 = 0, a_4 = 0, a_5 = 0, a_6 = -\frac{4}{9\pi}.$$

(13) H-3051 Dr Satish Kumar

So by Parseval's formula, we have

$$\int_0^L f^2(x) dx = \frac{c}{2} \left[\frac{a_0^2}{2} + a_1^2 + a_2^2 + a_3^2 + \dots \right]$$

$$\int_0^1 f^2(x) dx + \int_1^2 f^2(x) dx = \frac{2}{2} \left[\frac{\pi^2}{2} + 0 + \left(\frac{4}{\pi}\right)^2 + 0^2 + 0^2 + 0^2 + \left(\frac{-4}{9\pi}\right)^2 + \dots \right]$$

$$\Rightarrow \int_0^1 (\pi x)^2 dx + \int_1^2 \pi^2 (2-x)^2 dx = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \dots$$

$$\Rightarrow \frac{\pi^2}{3} + \pi^2 \left(- \frac{(2-x)^3}{3} \right) \Big|_1^2 = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \dots$$

$$\Rightarrow \frac{\pi^2}{3} - \pi^2 \left(0 - \frac{1}{3} \right) = \frac{\pi^2}{2} + \frac{16}{\pi^2} + \frac{16}{81\pi^2} + \dots$$

$$\Rightarrow \frac{2\pi^2}{3} - \frac{\pi^2}{2} = \frac{16}{\pi^2} \left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right]$$

$$\Rightarrow \frac{\pi^2}{6} = \frac{16}{\pi^2} \left[1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right]$$

$$\Rightarrow \frac{\pi^4}{96} = 1 + \frac{1}{3^4} + \frac{1}{5^4} + \dots \quad \text{Ans.}$$

(3) Prove that for $0 < x < \pi$

$$(a) x(\pi-x) = \frac{\pi^2}{6} - \left[\frac{\cos 2x}{1^2} + \frac{\cos 4x}{2^2} + \frac{\cos 6x}{3^2} + \dots \right]$$

$$(b) x(\pi-x) = \frac{8}{\pi} \left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \dots \right]$$

Deduce from (a) & (b) respectively

$$(c) \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \quad (d) \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}$$

(14) H-3051, Dr. Satish Kumar

Soln. Given $f(x) = x(\pi - x)$, $0 < x < \pi$

Half range cosine series is

$$f(x) = \frac{1}{2}a_0 + \sum a_n \cos nx$$

Where $a_0 = \frac{2}{\pi} \int_0^{\pi} f(x) dx = \frac{2}{\pi} \int_0^{\pi} x(\pi - x) dx$

$$= \frac{2}{\pi} \left[\pi \frac{x^2}{2} - \frac{x^3}{3} \right]_0^{\pi} = \frac{2}{\pi} \left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right]$$

$$a_0 = \frac{2}{\pi} \left[\frac{\pi^3}{6} \right] = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx dx = \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \cos nx dx$$

$$= \frac{2}{\pi} \left[(\pi x - x^2) \frac{\sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} (\pi - 2x) \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{1}{n} \left\{ (\pi - 2x) (-1) \frac{\cos nx}{n} \right\} \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} (-2) \frac{\cos nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[-\frac{1}{n} \left\{ \frac{\pi \cos n\pi}{n} + \frac{\pi}{n} \right\} - \frac{2}{n^2} (-\sin nx) \Big|_0^{\pi} \right]$$

$$a_n = \frac{2}{n} \left[-\frac{\cos n\pi}{n} - 1 \right] = -\frac{2}{n} \left[\frac{(-1)^n + 1}{n} \right]$$

$$a_n = -\frac{4}{n^2} \text{ if } n \text{ is even, } a_n = 0 \text{ if } n \text{ is odd.}$$

$$\therefore f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx \Rightarrow$$

$$\pi x - x^2 = \frac{1}{2} \left(\frac{\pi^2}{3} \right) + a_2 \cos 2x + a_4 \cos 4x + a_6 \cos 6x + \dots$$

$$\pi x - x^2 = \frac{\pi^2}{6} - 4 \left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \dots \right] \quad \left[\because a_n = 0 \text{ if } n \text{ is odd} \right]$$

$$\text{Parseval's formula} \Rightarrow \frac{2}{\pi} \int_0^{\pi} f(x)^2 dx = \frac{a_0^2}{2} + \sum a_n^2$$

$$\frac{2}{\pi} \int_0^{\pi} (\pi x - x^2)^2 dx = \frac{1}{2} \left(\frac{\pi^2}{6} \right)^2 + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

(15) H-3051, Dr. Satish Kumar

$$\frac{2}{\pi} \int_0^{\pi} (\pi^2 x^2 + x^4 - 2\pi x^3) dx = \frac{1}{2} \left(\frac{\pi^4}{9} \right)$$

$$+ 16 \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \right)$$

$$\Rightarrow \frac{2}{\pi} \left[\frac{\pi^2}{3} \cdot \pi^3 + \frac{1}{5} \cdot \pi^5 - \frac{2}{4} \pi \pi^4 \right] = \frac{\pi^4}{18} + 16 \left[\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \dots \right]$$

$$\Rightarrow \frac{2}{\pi} \left[\frac{\pi^5}{3} + \frac{\pi^5}{5} - \frac{1}{2} \pi^5 \right] = \frac{\pi^4}{18} + \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots \right]$$

$$\Rightarrow \frac{2\pi^5}{\pi} \left[\frac{1}{3} + \frac{1}{5} - \frac{1}{2} \right] = \frac{\pi^4}{18} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4}$$

$$\Rightarrow 2\pi^4 \left(\frac{10+6-15}{30} \right) = \frac{\pi^4}{18} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4}$$

$$\Rightarrow \frac{\pi^4}{15} = \frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{1}{n^4} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{\pi^4}{90}$$

$$\text{i.e. } \boxed{\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}}$$

(a) part is proved.

(b) Half range sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx dx$$

$$= \frac{2}{\pi} \int_0^{\pi} (\pi x - x^2) \sin nx dx$$

$$b_n = \frac{2}{\pi} \left[(\pi x - x^2) \left(-\frac{\cos nx}{n} \right) \Big|_0^{\pi} + \int_0^{\pi} (\pi - 2x) \frac{\cos nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[0 + \frac{(\pi - 2x) \cdot \sin nx}{n} \Big|_0^{\pi} - \int_0^{\pi} \frac{1}{n} \cdot (-2) \frac{\sin nx}{n} dx \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{n^2} \int_0^{\pi} \sin nx dx \right] = \frac{4}{\pi n^2} \left(-\frac{\cos nx}{n} \right) \Big|_0^{\pi}$$

$$b_n = -\frac{4}{\pi n^3} [(-1)^n - 1] \Rightarrow b_n = 0 \text{ if } n \text{ is even}$$

(16) H-3051 Dr. Satish Kumar

$$b_n = \frac{8}{n^3\pi} \text{ if } n \text{ is odd}$$

$$i. \pi x - x^2 = b_1 \sin x + b_3 \sin 3x + b_5 \sin 5x + \dots$$

[$\because b_n = 0$
if n is even]

$$\pi x - x^2 = \frac{8}{\pi} \left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \dots \right]$$

By Parseval's formula

$$\frac{2}{\pi} \int_0^\pi f^2(x) dx = \sum_{n=1}^{\infty} b_n^2, \quad n \text{ is odd}$$

$$\frac{2}{\pi} \int_0^\pi (\pi x - x^2)^2 dx = \sum_{n=1}^{\infty} b_n^2, \quad n \text{ is odd}$$

$$\frac{\pi^2}{15} = \left\{ \frac{8}{\pi} \left[\frac{1}{n^3} \right] \right\}^2, \quad n \text{ is odd}$$

$$\Rightarrow \frac{\pi^2}{15} = \sum_{n=1}^{\infty} \frac{64}{\pi^2} \cdot \frac{1}{n^6} \Rightarrow \frac{\pi^4}{15 \times 64} = \sum_{n=1}^{\infty} \frac{1}{n^6}, \quad n \text{ is odd}$$

$$\Rightarrow \frac{\pi^4}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots$$

$$\text{Let } S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \dots$$

$$S = \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \dots \right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \dots \right)$$

$$S = \frac{\pi^4}{960} + \frac{1}{2^6} \left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \dots \right)$$

$$S = \frac{\pi^4}{960} + \frac{S}{64} \Rightarrow \left(S - \frac{S}{64} \right) = \frac{\pi^4}{960}$$

$$\Rightarrow \frac{63}{64} S = \frac{\pi^4}{960}$$
$$\Rightarrow S = \frac{\pi^4 \times 64 \times 4}{63 \times 960 \times 60} \Rightarrow S = \frac{\pi^4}{945}$$